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A CLASSIFICATION OF MINIMAL SETS OF TORUS HOMEOMORPHISMS

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ABSTRACT. We provide a classification of minimal sets of homeomorphisms of the two-torus, in terms of the structure of their complement. We show that this structure is exactly one of the following types: (1) a disjoint union of topological disks, or (2) a disjoint union of essential annuli and topological disks, or (3) a disjoint union of one doubly essential component and bounded topological disks. Moreover, in case (1) bounded disks are non-periodic and in case (2) all disks are non-periodic.

This result provides a framework for more detailed investigations, and additional information on the torus homeomorphism allows to draw further conclusions. In the non-wandering case, the classification can be significantly strengthened and we obtain that a minimal set other than the whole torus is either a periodic orbit, or the orbit of a periodic circloid, or the extension of a Cantor set. Further special cases are given by torus homeomorphisms homotopic to an Anosov, in which types 1 and 2 cannot occur, and the same holds for homeomorphisms homotopic to the identity with a rotation set which has non-empty interior. If a non-wandering torus homeomorphism has a unique and totally irrational rotation vector, then any minimal set other than the whole torus has to be the extension of a Cantor set.

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1. Introduction and Statement of Results

As minimal sets relate naturally to many other dynamical notions, great effort has been devoted to the description of minimal sets and their intrinsic structure, which led to the identification of important subclasses like almost periodic or almost automorphic minimal sets (see, for example, [1, 22, 21] and references therein). However, there exist only very few situations in which a complete classification of the possible structure of minimal sets in a given manifold is available. One of the most important cases are orientation-preserving homeomorphisms of the circle, whose minimal sets classify into either periodic orbits, Cantor sets or the whole circle. By means of a suitable Poincaré section, this also provides a classification of minimal sets of flows on the two-torus generated by fixed point free vector fields, which are a suspension of one of the three types occurring for circle homeomorphisms. The Poincaré-Bendixon Theorem for planar flows or Aubry-Mather Theory for twist maps provide further classical examples (see e.g. [11]). More recently, homeomorphisms of the two-torus which are homotopic to the identity and have a single, totally irrational rotation vector were studied in [15]. The results in [15] include a classification of the minimal sets in terms of the structure of their complement. For general surface homeomorphisms, a more restricted classification is given in [2] under the additional *a priori* assumption of local connectedness. Here, our aim is to extend the main result in [15] to general homeomorphisms of the torus and to provide a strengthened classification for non-wandering torus homeomorphisms.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the two-dimensional torus, $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ the canonical projection and $\mathrm{Homeo}(\mathbb{T}^2)$ the set of homeomorphisms of $\mathbb{T}^2$. An open and connected set, respectively a compact and connected set, in the plane $\mathbb{R}^2$ or torus $\mathbb{T}^2$ is called a *domain*, respectively a *continuum*. We say an open set $D \subseteq \mathbb{T}^2$ is a *topological disk* if it is homeomorphic to $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and call it *bounded*, if the connected components of $\pi^{-1}(D)$ are bounded. Similarly, we say an open set $A \subseteq \mathbb{T}^2$ is a *(topological) annulus* if it is homeomorphic to the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ and call A *essential* if it contains a closed curve which is homotopically non-trivial in $\mathbb{T}^2$. We call an open set $B \subseteq \mathbb{T}^2$ *doubly essential* if it contains two homotopically nontrivial curves of different homotopy types. A subset A of

the torus (or any surface) is called a *circloid*, if it is contained in an embedded open annulus $\mathcal{A}$ and further (i) it is compact and connected, (ii) its complement in $\mathcal{A}$ consists of exactly two connected components $\mathcal{U}^-(A)$ and $\mathcal{U}^+(A)$ which are unbounded[1] below, respectively above, and (iii) it is minimal with respect to inclusion with properties (i) and (ii). A set which only satisfies (i) and (ii) is called an *annular continuum.* The homotopy type of A is defined as the homotopy type of an essential loop in $\mathcal{A}$. We call A essential if this homotopy type is non-zero and homotopically trivial otherwise. We call $A \subseteq \mathbb{T}^2$ *non-separating* if $A^c = \mathbb{T}^2 \setminus A$ is connected.

Circloids and annular continua appear frequently in the theory of torus and annular homeomorphisms [8, 9, 7, 13, 17, 10] and can be thought of as a generalisation of closed curves, adapted to the needs of topological dynamics. When f and $\widetilde{f}$ are homeomorphisms of the two-torus with minimal sets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, we say $(f, \mathcal{M})$ is an *extension* of $(\widetilde{f}, \widetilde{\mathcal{M}})$ if there exists a continuous onto map $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$, homotopic to the identity, which satisfies $\Phi \circ f = \widetilde{f} \circ \Phi$ and $\Phi(\mathcal{M}) = \widetilde{\mathcal{M}}$. When $\widetilde{\mathcal{M}}$ is finite we simply say $\mathcal{M}$ is an *extension of a periodic orbit*, when $\widetilde{\mathcal{M}}$ is a Cantor set we say $\mathcal{M}$ is an *extension of a Cantor set.*

Given a connected component U of $\mathcal{M}^c$ we say that it is periodic if there exists $n \in \mathbb{N}$ such that $f^n(U) = U$, otherwise we say that it is wandering. A minimal set of a homeomorphism f of the torus is a non-empty f-invariant compact set that is minimal, relative to inclusion, with respect to the properties of being f-invariant and compact. Our main result is the following.

Theorem 1 (Classification Theorem)**.** *Suppose* $f \in \mathrm{Homeo}(\mathbb{T}^2)$ *and* $\mathcal{M} \neq \mathbb{T}^2$ *is a minimal set. Then the complement of* $\mathcal{M}$ *consists of either:*

(1) *a disjoint union of topological disks.*
(2) *a disjoint union of at least one essential annulus and topological disks, where either:*
   (i) *the essential annuli in* $\mathcal{M}^c$ *are periodic and* $\mathcal{M}$ *is the orbit of the boundary of an essential periodic circloid, or*
   (ii) *every connected component in* $\mathcal{M}^c$ *is wandering and* f *is semi-conjugate to a one-dimensional irrational rotation,*
(3) *a disjoint union of exactly one doubly essential component and a number of bounded topological disks, where either:*
   (i) $\mathcal{M}$ *is an extension of a periodic orbit, or*
   (ii) $\mathcal{M}$ *is an extension of a Cantor set.*

Moreover, in case (1) bounded periodic disks cannot occur and in case (2) only essential annuli can be periodic.

We say a minimal set $\mathcal{M}$ is of type N with $N = 1, 2, 3$ if it belongs to case N in the above classification. This classification provides a basic framework for a more precise study of the different cases. In two important situations types 1 and 2 can be excluded. The first is the case where $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is homotopic to an Anosov homeomorphism on $\mathbb{T}^2$. Using classical results on Anosov homeomorphisms [3, 16, 23] one obtains the following.

Corollary 2. *Suppose* $f \in \mathrm{Homeo}(\mathbb{T}^2)$ *is homotopic to an Anosov homeomorphism. Then any minimal set of* f *is of type 3.*

The second situation is more intricate and concerns the case where f is homotopic to the identity. For such maps, an important topological invariant is the rotation set given by

$$\rho(F) \ = \ \left\{ \rho \in \mathbb{R}^2 \mid \exists z_i \in \mathbb{R}^2, \ n_i \nearrow \infty : \lim_{i \to \infty} \left(F^{n_i}(z_i) - z_i \right) / n_i = \rho \right\} \ , \tag{1.1}$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of f. This notion was introduced by Misiurewicz and Ziemian, who showed that $\rho(F)$ is always a compact and convex subset of the plane [19].

Corollary 3. *Suppose the rotation set of* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *has non-empty interior. Then any minimal set is of type 3.*

[1]Here, we identify $\mathcal{A}$ with $\mathbb{A}$ to define unboundedness.

A result of Misiurewicz and Ziemian [20] states that for all $\rho \in \mathrm{int}(\rho(F))$ there exists a minimal set M_ρ such that ρ is the unique rotation vector on M_ρ. In particular, there exist uncountably many minimal sets. Corollary 3 implies that for all non-rational ρ these are extensions of Cantor sets.

In the non-wandering case, a result of Koropecki [13] on aperiodic invariant continua of surface homeomorphisms allows to exclude unbounded disks in type 1 of the Classification Theorem. This leads to the following more restrictive classification. Recall that $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is called *non-wandering*, if there exist no wandering open sets.

Theorem 4 (Classification Theorem, non-wandering version)**.** *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is non-wandering and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set. Then one of the following holds:*

(1^{nw}) *$\mathcal{M}$ is a periodic orbit;*
(2^{nw}) *$\mathcal{M}$ is the orbit of a periodic circloid;*
(3^{nw}) *$\mathcal{M}$ is the extension of a Cantor set, with all connected components non-separating.*

Note that (1^{nw}) and (3^{nw}) belong to type (3) in Theorem 1. (2^{nw}) belongs either to (2) or (3), depending on whether the circloid is essential or not, since the orbit of a homotopically trivial periodic circloid is a periodic orbit extension.

Further information can be deduced if the rotation set of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is reduced to a single point. In this case, we call f a *pseudo-rotation.*

Corollary 5. *Suppose f is a non-wandering pseudo-rotation with rotation vector ρ and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set.*

(a) *If ρ is totally irrational (its coordinates are rationally independent), then $\mathcal{M}$ is an extension of a Cantor set.*
(b) *If ρ is rational, then $\mathcal{M}$ is either an extension of a Cantor set, or the periodic orbit of either a point or a homotopically trivial circloid.*

The paper is organised as follows. In Section 2, we collect several preliminary topological results which will be used in the later sections. In particular, we describe a procedure to fill in subsets of the torus, similar to a standard construction in the plane. Section 3 then contains the proof of the main classification. In Section 4, we consider several special cases and applications of the classification. Finally, in Section 5 we list and discuss a number of further problems that naturally arise from the results in this paper.

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2. Fill-in Constructions

In this section we collect several topological facts which are later mixed with dynamical arguments to obtain our main results. In particular, we describe a procedure to *'fill in'* subsets of the torus which is similar to a standard construction in the plane, but requires take care of some subtleties of surface topology. Even though some of these constructions may be considered folklore, we therefore spell out the details.

2.1. **Notation.** Given a metric space (X, d) and $C, D \subset X$, the Hausdorff distance is defined as

$$d_{\mathcal{H}}(C, D) = \max\{\sup_{x \in C} d(x, D), \sup_{y \in D} d(y, C)\}. \tag{2.1}$$

The convergence of a sequence $\{C_n\}_{n\in\mathbb{N}}$ of subsets in X to $A \subset X$ in this distance is denoted either by $C_n \to_{\mathcal{H}} A$ or by $\lim^{\mathcal{H}}_{n\to\infty} C_n = A$. Note that $d_{\mathcal{H}}(C, D) < \varepsilon$ if and only if $C \subseteq B_\varepsilon(D)$ and $D \subseteq B_\varepsilon(C)$, and that the Hausdorff distance defines a metric if one restricts to compact subsets.

The fundamental group of $\mathbb{T}^2$ will be denoted by $\pi_1(\mathbb{T}^2)$. Given a domain $U \subseteq \mathbb{T}^2$, consider the subgroup G in $\pi_1(\mathbb{T}^2)$ given by classes of loops (i.e. simple closed curves) in U. We say that U is *homotopically trivial* if $G = \{0\}$, *essential* if G is isomorphic to $\mathbb{Z}$ and *doubly essential* if G is isomorphic to $\mathbb{Z}^2$. To an essential set $U \subset \mathbb{T}^2$ we can associate a homotopy type given by a vector $(p, q) \in \mathbb{Z}^2$ with $\gcd(p, q) = 1$, where (p, q) is the generator of G. In this case, we call U a (p, q)-essential set. It is verified that an essential set U is (p, q)-essential if and only if every connected component $\widetilde{U}$ of its lift satisfies $\widetilde{U} + (p, q) = \widetilde{U}$. We use $\mathrm{Conn}(U)$ to denote the set of connected components of $U \subseteq \mathbb{T}^2$ and $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ to denote the Riemann sphere. We call a connected set $A \subset \mathbb{T}^2$ *bounded*, if all connected components of $\pi^{-1}(A)$ are bounded. Note that this does not imply a uniform bound on the size of the connected components of $\pi^{-1}(A)$ (see Remark 6).

Finally, given a homotopically trivial simple loop $\gamma \subseteq \mathbb{T}^2$, define the embedded Jordan disk $B(\gamma) := \pi(B(\gamma_0)) \subset \mathbb{T}^2$, where $\gamma_0 \subset \mathbb{R}^2$ is a lift of γ and $B(\gamma_0)$ the Jordan disk bounded by γ_0.

2.2. **Fill-in of planar sets.** Given any connected set $A \subseteq \mathbb{R}^2$, let $U_\infty(A)$ be the connected component of $\overline{\mathbb{R}^2} \setminus A$ which contains the point ∞. The standard way to fill in the set A is to define

$$\mathrm{Fill}_{\mathbb{R}^2}(A) \ = \ \mathbb{R}^2 \setminus U_\infty(A) \ . \tag{2.2}$$

Note that when γ is a loop in $\mathbb{R}^2$, then $\mathrm{Fill}_{\mathbb{R}^2}(\gamma)$ is just the closure of the Jordan domain of γ, which will be denoted by $B(\gamma)$. Equivalent definitions of $\mathrm{Fill}_{\mathbb{R}^2}(A)$ are the following. First, if $\{A_\alpha\}_{\alpha \in I}$ is the set of bounded connected components of $\mathbb{R}^2 \setminus A$, then

$$\mathrm{Fill}_{\mathbb{R}^2}(A) := A \cup \bigcup_{\alpha \in I} A_\alpha \tag{2.3}$$

Since the union of a connected set with a connected component of its complement is connected[2], this allows to see in particular that $\mathrm{Fill}_{\mathbb{R}^2}(A)$ is always connected. Secondly, if we say that a set $A \subseteq \mathbb{R}^2$ is *filled-in* if $\overline{\mathbb{R}^2} \setminus A$ is connected, then $\mathrm{Fill}_{\mathbb{R}^2}(A)$ is just the smallest filled-in set that contains A. For domains, a third equivalent characterisation is given by the first part of the next statement.

Lemma 2.1. *Let $A_0 \subseteq \mathbb{R}^2$ be open and connected. Then*

$$\mathrm{Fill}_{\mathbb{R}^2}(A_0) = \{z \in \mathbb{T}^2 \mid \exists a \text{ loop } \gamma \subseteq A_0 : z \in B(\gamma)\}. \tag{2.4}$$

Further, for all $v \in \mathbb{R}^2$ we have that $A_0 \cap (A_0 + v) = \emptyset$ implies $\mathrm{Fill}_{\mathbb{R}^2}(A_0) \cap (\mathrm{Fill}_{\mathbb{R}^2}(A_0) + v) = \emptyset$.

Proof. Let

$$\widetilde{A}_0 \ = \ \{z \in \mathbb{R}^2 \mid \exists \text{a loop } \gamma \subseteq A_0 : z \in B(\gamma)\} \ . \tag{2.5}$$

Then $\widetilde{A}_0$ is simply-connected and therefore a topological disk by the Riemann Mapping Theorem. In particular, it is filled-in. Now, suppose B is another filled-in set that contains A_0, but does not contain $\widetilde{A}_0$. Then there is a loop in A_0 such that its bounded component contains a point that is not in B. However, this point cannot belong to the unbounded component of $\mathbb{R}^2 \setminus A$, which is a contradiction. It follows that any filled-in set that contains A_0 also contains $\widetilde{A}_0$, and therefore $\widetilde{A}_0 = \mathrm{Fill}_{\mathbb{R}^2}(A_0)$. The second statement is a consequence of the first statement. □

Our aim is now to define a similar Fill-operation for connected subsets of the torus. This does not work for arbitrary connected subsets of the torus (see the remarks at the end of this section), but we show it does apply to subsets of $\mathbb{T}^2$ which are either domains or bounded continua. At the end of this section we collect several basic results that will be used later in the proof of our main results.

[2]This is true in any σ-compact connected Hausdorff space.

2.3. **Fill-in of domains in the torus.** We say that a domain $A \subseteq \mathbb{T}^2$ is *locally homotopically trivial* if every loop contained in A which is homotopically trivial in $\mathbb{T}^2$ is also homotopically trivial in A. Note that for instance an essential annulus is locally homotopically trivial.

Lemma 2.2. *If a domain $A \subseteq \mathbb{T}^2$ is locally homotopically trivial, then any connected component of $\pi^{-1}(A) \subset \mathbb{R}^2$ is simply connected.*

Proof. Let $A_0 \subset \mathbb{R}^2$ be a connected component of $\pi^{-1}(A)$ and let $\gamma_0 \subset A_0$ be a simple closed curve. We have to show that A_0 contains $B(\gamma_0)$. Since A_0 is open, by approximating γ_0 by an analytic curve homotopic to γ_0, we may as well assume that that $\gamma := \pi(\gamma_0)$ has finitely many self-intersections. Consequently, since γ_0 is compact, only finitely many other integer translates of γ_0 intersect γ_0, and the number of intersection points of γ_0 with the integer translates of γ_0 is finite. Therefore, the intersection pattern produces a finite number of Jordan disks $J_1, \ldots, J_n$ such that the boundary of J_i is contained in $\pi^{-1}(\gamma)$, $J_i \cap \pi^{-1}(\gamma) = \emptyset$ and $\overline{B(\gamma_0)} = \bigcup_{i=1}^n \overline{J_i}$. Further, each of the disks J_i embeds injectively in $\mathbb{T}^2$, since otherwise we would have an intersection $J_i \cap (J_i + v)$ for some $i \in \{1, \ldots, n\}$, $v \in \mathbb{Z}^2$, contradicting the definition of the J_i. Since A is simply connected and $\partial\pi(J_i) \subseteq A$, we obtain $B(\pi(J_i)) \subseteq A$. However, this implies $J_i \subseteq A_0$ for all $i = 1, \ldots, n$ and therefore $B(\gamma_0) \subseteq A_0$. □

Locally homotopically trivial domains in the torus in the above sense can be classified as follows.

Lemma 2.3 ([15], Lemma 7)**.** *Let $A \subset \mathbb{T}^2$ be open, trivial (respectively essential, doubly essential) and simply connected. Then A is a disk (respectively essential annulus, $\mathbb{T}^2$).*

Given a domain $A \subseteq \mathbb{T}^2$ and a connected component A_0 of its lift, define

$$\mathrm{Fill}(A) \;:=\; \pi(\mathrm{Fill}_{\mathbb{R}^2}(A_0)). \tag{2.6}$$

Note that this definition does not depend on the choice of connected component of $\pi^{-1}(A)$. Let us collect basic properties of the Fill-operation for domains.

Proposition 2.4 (Fill-in of torus domains)**.** *Suppose $A \subseteq \mathbb{T}^2$ is a domain. Then the following hold.*

- (a) $\mathrm{Fill}(A) = \{z \in \mathbb{T}^2 \mid \exists a \text{ trivial loop } \gamma \subseteq A : z \in B(\gamma)\}$.
- (b) *A is trivial and bounded iff* $\mathrm{Fill}(A)$ *is a bounded disk.*
- (c) *A is trivial and unbounded iff* $\mathrm{Fill}(A)$ *is an unbounded disk.*
- (d) *A is (p, q)-essential iff* $\mathrm{Fill}(A)$ *is a (p, q)-annulus.*
- (e) *A is doubly essential iff* $\mathrm{Fill}(A) = \mathbb{T}^2$.
- (f) $\partial\mathrm{Fill}(A) \subseteq \partial A$.
- (g) *If $f \in \mathrm{Homeo}(\mathbb{T}^2)$, then* $f(\mathrm{Fill}(A)) = \mathrm{Fill}(f(A))$.

Proof. The property (a) is a direct consequence of Lemma 2.1 combined with Lemma 2.2 applied to a connected component A_0 of $\pi^{-1}(A)$. As a consequence, we obtain that $\mathrm{Fill}(A)$ is simply-connected in $\mathbb{T}^2$, such that according to Lemma 2.3 it is either a disk, an essential annulus or a doubly essential set. The properties (b)–(e) therefore follow from the second part of Lemma 2.1.

To prove (f), assume for a contradiction that $z \in \partial\mathrm{Fill}(A)$ but $z \notin \partial A$. Then, since $A \subseteq \mathrm{Fill}(A)$ is open, we have $z \notin \mathrm{Cl}[A]$. Let z_0 be a lift of z and denote by C the connected component of z_0 in $\mathbb{R}^2 \setminus \pi^{-1}(\mathrm{Cl}[A])$. Then either C is contained in some integer translate of $\mathrm{Fill}(A_0)$, but then $z \subseteq \pi(C) \subseteq \mathrm{int}(\mathrm{Fill}(A))$, or C is disjoint from $\pi^{-1}(\mathrm{Fill}(A))$, but then $z \in \pi(C) \subseteq \mathrm{int}(\mathbb{T}^2 \setminus \mathrm{Fill}(A))$. Hence, in both cases we arrive at a contradiction.

Finally, in order to show (g) let $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $f \in \mathrm{Homeo}(\mathbb{T}^2)$. We then have

$$f(\mathrm{Fill}(A)) = f \circ \pi(\mathrm{Fill}_{\mathbb{R}^2}(A_0)) = \pi \circ F(\mathrm{Fill}_{\mathbb{R}^2}(A_0)) \overset{(a)}{=} \pi(\mathrm{Fill}_{\mathbb{R}^2}(F(A_0))) = \mathrm{Fill}(A). \tag{2.7}$$

This finishes the proof. □

2.4. **Fill-in of continua in the torus.** We now proceed to construct the fill-in of bounded continua in the torus. As the following remark shows, some subtleties have to be addressed and the construction does not work for general subsets of $\mathbb{T}^2$.

Remark 6. (1) If $A \subset \mathbb{T}^2$ is a continuum, but not bounded, it is not clear how to define a fill-in. For example, consider the disjoint union of two essential loops $\gamma_1, \gamma_2 \subseteq \mathbb{T}^2$ with an infinite embedded line $\gamma \subseteq \mathbb{T}^2$ that accumulates on γ_1 in one and on γ_2 in the other direction. Then the complement of any lift of $\gamma_1 \cup \gamma_2 \cup \gamma$ to $\mathbb{R}^2$ will have three connected components, and $\mathrm{Fill}(\gamma_1 \cup \gamma_2 \cup \gamma)$ will depend on the particular choice and position of these components in the plane.

(2) A second problem comes from the fact that even if a connected subset of $A \subseteq \mathbb{T}^2$ is bounded in the sense of Section 2.1, there is not necessarily a uniform bound on the diameter of the connected components of $\pi^{-1}(A)$. Indeed, consider an irrational foliation of the torus given by the orbits of a Kronecker flow. For each $n \in \mathbb{N}$ let A_n be a segment of length n in one of the leaves of this foliation, chosen such that no two segments are in the same leave. Then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that A_n is ε-dense. This implies that $A = \bigcup_{n\in\mathbb{N}} A_n$ is connected: Suppose $\{U, V\}$ is an open disjoint cover of A. Then for $\varepsilon > 0$ sufficiently small both U and V contain a disk of radius ε, and consequently A_n intersects both U and V when n is large. Since $A_n \subseteq A \subseteq U \cup V$, this contradicts the connectedness of A_n. The connected components of $\pi^{-1}(A)$ are exactly the connected components of the lifts of the segments A_n. Hence, A is bounded according to the above definition, but the connected components of $\pi^{-1}(A)$ are not uniformly bounded in diameter.

In order to avoid problems, we restrict to bounded continua and first show that their complement is always doubly essential. This implies immediately that for any bounded continuum $A \subseteq \mathbb{T}^2$ there is only one connected component of $\pi^{-1}(A)$ up to translation by integer vectors. As a consequence, we will be able to define the fill-in in the same way as for domains.

In what follows, for a given family of pairwise disjoint sets $\{X_n\}_{n\in\mathbb{N}}$ either in the plane or the torus, we denote its union by $\biguplus_{n\in\mathbb{N}} X_n$.

Lemma 2.5. *Let $\{D_n\}_{n\in\mathbb{N}}$ be a family of pairwise disjoint bounded open disks in the plane. Then, $A = \mathbb{R}^2 \setminus \biguplus_{n\in\mathbb{N}} D_n$ is a connected set.*

Proof. In order to prove the connectedness of A, we show that given any two points $z_0, z_1 \in A$ there exists a connected subset of A containing z_0 and z_1. Denote the straight line segment from z_0 to z_1 by S. We assume without loss of generality that $S = [0, 1] \times \{0\}$ and equip it with the canonical order on the unit interval. Further, define

$$N = \{n \in \mathbb{N} \mid D_n \cap S \neq \emptyset\} \text{ and } C = (S \cap A) \cup \bigcup_{n\in N} \partial D_n. \tag{2.8}$$

We claim that C is connected. In order to see this, suppose for a contradiction that $U, V \subseteq \mathbb{R}^2$ are disjoint open sets which both intersect C and whose union covers C. Suppose $z_0 \in U$ and let $z_1' \in C \cap V$. Further, let $z_- = \sup\{z \in S \cap A \cap U \mid z \leq z_1'\}$. By compactness, $z_- \in S \cap A \cap V^c = S \cap A \cap U$. Consequently, z_- is the left endpoint of an interval $I = S \cap D_n \subseteq S \setminus A$ for some $n \in \mathbb{N}$, and the right endpoint z_+ of this interval belongs to $S \cap A \cap V$. However, this means that U and V both intersect ∂D_n, contradicting the connectedness of ∂D_n. □

Using this together with Proposition 2.4, we can now show that compact and bounded subsets of the torus have doubly essential complement.

Lemma 2.6. *If $A \subseteq \mathbb{T}^2$ is compact and bounded, then A^c is doubly essential. Consequently, if A is connected, if A is connected, all connected components of $\pi^{-1}(A)$ project injectively onto A and coincide up to translation by an integer vector.*

Proof. Suppose that A^c is not doubly essential. Then due to Proposition 2.4, for every $U \in \mathrm{Conn}(A^c)$ the set $\mathrm{Fill}(U)$ is either a bounded or unbounded topological disk or an

essential annulus. We distinguish between three corresponding cases and show that in each of them $\pi^{-1}(A)$ contains an unbounded connected component, contradicting the boundedness of A.

First assume that there exists $U \in \mathrm{Conn}(A^c)$ such that $\mathrm{Fill}(U)$ is an essential annulus. Then $\partial\mathrm{Fill}(U)$ consists of one or two connected components, which are both contained in A but at the same time lift to unbounded components in $\mathbb{R}^2$. Secondly, suppose there exists $U \in \mathrm{Conn}(A^c)$ such that $D = \mathrm{Fill}(U)$ is an unbounded disk. Fix a connected component $D_0 \subseteq \pi^{-1}(D)$ and a point $z_0 \in \partial D_0 \subseteq \pi^{-1}(A)$. Then for any $N \geq 0$ we can choose a sequence $\eta_n \subseteq D_0$ of arcs of diameter N converging in Hausdorff topology and such that $\eta := \lim^{\mathcal{H}}_{n\to\infty} \eta_n$ contains z_0 and is contained in ∂D_0. For example, we can identify D_0 with $\mathbb{D}$ by the Riemann Mapping Theorem and choose the η_n suitable segments in the circles of radius $1-1/n$. Then η is connected (as the Hausdorff limit of connected sets) and of diameter N. Since N was arbitrary, this shows that the connected component of z_0 in $\partial D_0 \subseteq \pi^{-1}(A)$ is unbounded. Finally, assume that all connected components of A^c are bounded. Then

$$\widetilde{A}_0 = \mathbb{R}^2 \setminus \bigcup_{U\in\mathrm{Conn}(\mathbb{R}^2\setminus\pi^{-1}(A))} \mathrm{Fill}_{\mathbb{R}^2}(U)$$

is the complement of a family of bounded disks and therefore connected and unbounded by Lemma 2.5.

Thus, as claimed the complement of A contains a doubly essential component. This implies that A is contained in a bounded topological disk D. Every connected component D_0 of $\pi^{-1}(D)$ therefore contains a subset A_0 that projects injectively onto A. Since $\pi : D_0 \to D$ is a homeomorphism we obtain that A_0 is connected. □

Given a bounded continuum $A \subseteq \mathbb{T}^2$, we now choose an arbitrary connected component A_0 of $\pi^{-1}(A)$ and let

$$\mathrm{Fill}(A) \ := \ \pi(\mathrm{Fill}_{\mathbb{R}^2}(A_0)) \ . \tag{2.9}$$

Since all connected components of $\pi^{-1}(A)$ coincide up to integer translations, this definition does not depend on the choice of A_0. Furthermore, we have the following.

Lemma 2.7. *Suppose A is a bounded continuum. Then*

$$\mathrm{Fill}(A) = A \cup \biguplus_{n\in\mathcal{N}} D_n(A), \tag{2.10}$$

where the $D_n(A)$ are bounded disks and

$$\partial D_n(A) = A \cap \mathrm{Cl}[D_n(A)] \tag{2.11}$$

for every $n \in \mathcal{N} \subseteq \mathbb{N}$.

Proof. Let A_0 be a connected component of $\pi^{-1}(A)$ and $\{D_n(A_0)\}_{n\in\mathcal{N}}$ be the bounded connected components of $\mathbb{R}^2 \setminus A_0$. As A is connected, $D_n(A_0)$ is a disk for every $n \in \mathcal{N}$. Further, we have

$$\partial D_n(A_0) = A_0 \cap \mathrm{Cl}[D_n(A_0)] \tag{2.12}$$

and since A_0 is bounded we have that

$$D_n(A_0) \cap (D_n(A_0) + v) = \emptyset \tag{2.13}$$

for every $v \in \mathbb{Z}^2 \setminus \{0\}$. Consequently $\pi(D_n(A_0)) = D_n(A)$ is a bounded disk for every $n \in \mathcal{N}$. We obtain that

$$\mathrm{Fill}(A) = \pi(\mathrm{Fill}(A_0)) = \pi(A_0 \cup \biguplus_{n\in\mathcal{N}} D_n(A_0)) = A \cup \biguplus_{n\in\mathcal{N}} D_n(A) \tag{2.14}$$

and

$$\partial D_n(A) = \partial\pi(D_n(A_0)) = \pi(\partial D_n(A_0)) = \pi(A_0 \cap \mathrm{Cl}[D_n(A_0)]) = A \cap \mathrm{Cl}[D_n(A)]. \tag{2.15}$$

This finishes the proof. □

3. Classification of minimal sets

In this section, we prove the main classification given in the introduction.

3.1. **Proof of the Classification Theorem.** We start the proof with the following trichotomy.

Proposition 3.1. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set. Then one of the following holds:*

(1) *$\mathcal{M}^c$ is a disjoint union of topological disks.*
(2) *$\mathcal{M}^c$ is a disjoint union of one or more essential annuli and topological disks.*
(3) *$\mathcal{M}^c$ is a disjoint union of one double essential component and bounded topological disks.*

Proof. Let $\mathrm{Conn}_T(\mathcal{M}^c) \subset \mathrm{Conn}(\mathcal{M}^c)$ be the trivial connected components and $\mathrm{Conn}_E(\mathcal{M}^c)$ the essential connected components of $\mathcal{M}^c$. Consider

$$\mathcal{M}' := \mathbb{T}^2 \setminus \bigcup_{\Sigma \in \mathrm{Conn}_T(\mathcal{M}^c)} \mathrm{Fill}(\Sigma), \tag{3.1}$$

which is compact and f-invariant by Proposition 2.4(g). We claim that $\mathcal{M}' \cap \mathcal{M} \neq \emptyset$. In order to see this, let

$$\mathcal{F} = \{\mathrm{Fill}(\Sigma) \mid \Sigma \in \mathrm{Conn}_T(\mathcal{M}^c)\} \tag{3.2}$$

and

$$\mathcal{D} = \left\{ \bigcup_{n \in \mathbb{N}} F_n \mid F_n \text{ is an increasing sequence in } \mathcal{F} \right\}. \tag{3.3}$$

First, note that all elements in $\mathcal{F}$ are topological disks with boundary contained in $\mathcal{M}$. Further, $\mathcal{F}$ is partially ordered by inclusion and two elements of $\mathcal{F}$ are either disjoint or one is contained in the other. Consequently, the same is true for $\mathcal{D}$. Furthermore, let $\{F_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{F}$. Then $D = \bigcup_{n\in\mathbb{N}} F_n \in \mathcal{D}$ is a disk, and

$$\lim^{\mathcal{H}}_{n\to\infty} \partial F_n = \partial D \subseteq \mathcal{M}. \tag{3.4}$$

Now, suppose $\{D_n\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{D}$. Then $\widehat{D} = \bigcup_{n\in\mathbb{N}} D_n$ is again an element of $\mathcal{D}$. Hence, if we define $\mathcal{D}_{\max}$ as the set of maximal elements of $\mathcal{D}$, then due to the Lemma of Zorn every element of $\mathcal{D}$ is contained in an element of $\mathcal{D}_{\max}$. Thus, we have

$$\mathcal{M}' \;=\; \mathbb{T}^2 \setminus \bigcup_{D \in \mathcal{D}} D \;=\; \mathbb{T}^2 \setminus \biguplus_{D \in \mathcal{D}_{max}} D\;.$$

Since $\partial D \subset \mathcal{M}$ for every $D \in \mathcal{D}_{max}$, this implies in particular that $\mathcal{M}' \cap \mathcal{M} \neq \emptyset$. By minimality of $\mathcal{M}$ we therefore have $\mathcal{M} \subset \mathcal{M}'$, so that $\mathrm{Fill}(\Sigma) = \Sigma$ for every $\Sigma \subset \mathrm{Conn}_T(\mathcal{M}^c)$. In other words, every trivial component of $\mathcal{M}^c$ is a disk.

There exists an integer vector $(p,q) \in \mathbb{Z}^2 \setminus \{0\}$ such that every element in $\mathrm{Conn}_E(\mathcal{M}^c)$ is (p,q)-essential. Moreover, the above argument adapted to this case shows that for every $\Sigma \in \mathrm{Conn}_E(\mathcal{M}^c)$ we have $\mathrm{Fill}(\Sigma) = \Sigma$, and hence every $\Sigma \in \mathrm{Conn}_E(\mathcal{M}^c)$ is a (p,q)-annulus.

To conclude the proof, it now suffices to remark that unbounded disks or essential components cannot coexist with a doubly essential component, and that any doubly essential component is necessarily unique. □

In order to show the non-existence of periodic bounded disks in $\mathrm{Conn}(\mathcal{M}^c)$, we start with a preliminary lemma.

Lemma 3.2. *Let $\mathcal{M} \subset \mathbb{T}^2$ be a connected minimal set of a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$. Further, assume that there exists a periodic bounded disk $D_0 \in \mathrm{Conn}(\mathcal{M}^c)$ of period p. Then, $\mathcal{M} = \partial D_0 = \ldots = \partial D_{p-1}$, where $D_k = f^k(D_0)$ for $k = 1, \ldots, p-1$.*

Proof. Since $\partial D_0 \cup ... \cup \partial D_{p-1}$ is an f-invariant set contained in $\mathcal{M}$, we have that

$$\mathcal{M} = \partial D_0 \cup ... \cup \partial D_{p-1}.$$

Given $x \in \mathcal{M}$, let $r(x) \in \{1, ..., p\}$ be the number of disks in $\{D_0, ..., D_{p-1}\}$ for which $x \in \mathrm{Cl}[D_k]$ $(k = 0, ..., p-1)$. Now, for any $k_0 = 0, \ldots, p-1$ the set $r^{-1}(\{k : k \geq k_0\}) \subseteq \mathcal{M}$ is closed and invariant, and therefore either empty or equal to $\mathcal{M}$. By minimality of $\mathcal{M}$, this implies that r is constant, say $r = m \in \{1, \ldots, p\}$.

Now, for every $x \in \mathcal{M}$ define $I_x = \bigcap_{i=1}^{m} \partial D_{k_i}$, where $D_{k_1}, ..., D_{k_m}$ are the disks for which $x \in \mathrm{Cl}[D_{k_i}]$. For every $x \in \mathcal{M}$ the set I_x is closed, and the collection of these sets $\mathcal{Y} = \{I_x : x \in \mathcal{M}\}$ is a finite family. Suppose that it is given by $I_{x_1}, ..., I_{x_N}$. Then, we have that $\mathcal{M} = \bigcup_{i=1}^{N} I_{x_i}$. Further, $I_{x_i} \cap I_{x_j} = \emptyset$ if $i \neq j$, since $z \in I_{x_i} \cap I_{x_j}$ would imply $r(z) > m$. Therefore, by connectedness of $\mathcal{M}$ we have

$$I_{x_1} = ... = I_{x_N} = \mathcal{M},$$

which implies that

$$\mathcal{M} = \partial D_0 = ... = \partial D_{p-1}.$$

This finishes the proof. □

Lemma 3.3. *Let $\mathcal{M} \subset \mathbb{T}^2$ be a connected minimal set of a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, and assume that* $\mathrm{Conn}(\mathcal{M}^c)$ *does not contain any doubly essential component. Then every bounded disk $D_0 \in \mathrm{Conn}(\mathcal{M}^c)$ is non periodic.*

Proof. Let us suppose for a contradiction that there exists a periodic bounded disk $D_0 \in \mathrm{Conn}(\mathcal{M}^c)$. Since $\mathcal{M}$ is connected, Lemma 3.2 shows that $\mathcal{M} = \partial D_0$. Let $\widetilde{D}_0$ be a connected component of $\pi^{-1}(D_0)$. Then $\pi : \widetilde{D}_0 \to D_0$ is a homeomorphism and $\pi : \partial\widetilde{D}_0 \to \partial D_0$ is onto. We now split the proof into two cases, each leading to a contradiction. First suppose

$$\left(\mathrm{Cl}[\widetilde{D}_0] + v\right) \cap \mathrm{Cl}[\widetilde{D}_0] = \emptyset \text{ for every } v \in \mathbb{Z}^2 \setminus \{0\}.$$

Then $\mathrm{Cl}[D_0]$ is bounded and compact, such that Lemma 2.6 implies the existence of a doubly essential component in D_0^c. Secondly, assume that

$$\left(\mathrm{Cl}[\widetilde{D}_0] + v\right) \cap \mathrm{Cl}[\widetilde{D}_0] \neq \emptyset \text{ for some } v \in \mathbb{Z}^2 \setminus \{0\}.$$

In this case, define $r : \mathcal{M} \to \mathbb{Z}$ as $r(x) = \#\{v \in \mathbb{Z}^2 \setminus \{0\} : \widetilde{x} + v \in \partial\widetilde{D}_0\}$ where $\widetilde{x} \in \partial\widetilde{D}_0 \cap \pi^{-1}(x)$. It is verified that $r(x)$ does not depend on $\widetilde{x}$. Since $\widetilde{D}_0$ is bounded, r is finite. Further, we have that $r^{-1}(\{k : k \geq k_0\})$ is a closed and f-invariant subset of $\mathcal{M}$ for every $k_0 \in \mathbb{Z}$. This implies by minimality of $\mathcal{M}$ that $r^{-1}(\{k : k \geq k_0\})$ is either empty or equal to $\mathcal{M}$, so $r(x)$ does not depend on $x \in \mathcal{M}$. Therefore, $r(x) = m$ for some positive integer m. Define

$$\mathcal{Y} = \left\{(v_1, ..., v_m) \in (\mathbb{Z}^2)^m : \exists z \in \partial\widetilde{D}_0 \text{ such that } z + v_1, ..., z + v_m \in \partial\widetilde{D}_0\right\}.$$

Since $\widetilde{D}_0$ is bounded, the set $\mathcal{Y}$ has to be finite, say $\mathcal{Y} = \{\xi_1, ..., \xi_N\}$ with $\xi_i = (v_1^i, \ldots, v_m^i)$. For $k = 1, ..., N$ define the sets

$$A_k = \{z \in \partial\widetilde{D}_0 : z + v_1^k, ..., z + v_m^k \in \partial\widetilde{D}_0\}.$$

It is readily verified that $A_k \subset \partial\widetilde{D}_0$ is closed and that $\partial\widetilde{D}_0 = \bigcup_{k=1}^{N} A_k$.

When ξ_i is just a permutation of the vector ξ_j, then obviously $A_i = A_j$. Otherwise, we must have $A_i \cap A_j = \emptyset$, since in this case the value $r(z)$ would be strictly greater than m for any $z \in A_i \cap A_j$, which is not possible. Therefore the sets A_i are either equal or pairwise disjoint. As $\partial\widetilde{D}_0$ is connected all sets A_i have to coincide, and this implies $\partial\widetilde{D}_0 = A_1$. However, this means that $z + nv_j^1 \in \partial\widetilde{D}_0$ for every $n \in \mathbb{N}$ and $j = 1, \ldots, m$, contradicting the boundedness of $\widetilde{D}_0$. □

Proposition 3.4. *If $\mathcal{M}$ in Proposition 3.1 is of type 1 or 2, then* $\mathrm{Conn}(\mathcal{M}^c)$ *does not contain any bounded periodic disk.*

Proof. Suppose for a contradiction that $\mathcal{M}$ is of type 1 or 2 and $D_0 \in \mathrm{Conn}(\mathcal{M}^c)$ is a bounded periodic disk of period p. Let $D_k = f^k(D_0)$ as before. Then $\bigcup_{k=0}^{p-1} \partial D_k \subseteq \mathcal{M}$ is compact and invariant, so that by minimality $\bigcup_{k=0}^{p-1} \partial D_k = \mathcal{M}$.

Let Λ_0 be the connected component of $\mathcal{M}$ which contains ∂D_0. Then Λ_0 is q-periodic for some $q \leq p$ and minimal for f^q. By Lemma 3.3, Λ_0^c contains a doubly essential component, and so does $f^i(\Lambda_0)^c$ for $i = 0, \ldots, q-1$. However, due to Lemma 2.6 this implies that

$$\mathcal{M}^c = \left(\bigcup_{i=0}^{q-1} f^i(\Lambda_0) \right)^c \tag{3.5}$$

contains a doubly essential component, contradicting our assumption. □

In the next section we will see that also unbounded disk are wandering for type 2 minimal sets.

3.2. **Minimal sets of type 2.** In this section, we give a more detailed description of minimal sets of type 2. Our aim is the following addendum to Proposition 3.1.

Addendum 3.5. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}$ is a minimal set of type 2. Then one of the following holds.*

(i) *The essential annuli in Conn($\mathcal{M}^c$) are periodic and $\mathcal{M}$ is the orbit of the boundary of an essential periodic circloid. Further any disk in Conn($\mathcal{M}^c$) is wandering.*

(ii) *f is semiconjugate to a one-dimensional irrational rotation, and every element in Conn($\mathcal{M}^c$) is wandering.*

We start with some purely topological facts concerning circloids. We call a set $A \subseteq \mathbb{A}$ *essential*, if $\mathbb{A} \setminus A$ does not contain a connected component which is unbounded above and below. If A is bounded above, we denote by $\mathcal{U}^+(A)$ the connected component of $\mathbb{A} \setminus \mathrm{Cl}(A)$ which is unbounded above. Similarly, we define $\mathcal{U}^-(A)$ when A is bounded below. Further, we write $\mathcal{U}^{-+}(A)$ instead of $\mathcal{U}^+(\mathcal{U}^-(A))$, and use analogous notation for longer concatenations of these operations. This leads to a simple procedure to produce circloids.

Lemma 3.6 ([10])**.** *Suppose $A \subseteq \mathbb{A}$ is essential and bounded above. Then*

$$\mathcal{C}^+(A) \;=\; \mathbb{A} \setminus (\mathcal{U}^{+-}(A) \cup \mathcal{U}^{+-+}(A)) \tag{3.6}$$

is a circloid. Further $\mathcal{U}^{+-+-}(A) = \mathcal{U}^{+-}(A)$.

We call $\mathcal{C}^+(A)$ the *upper frontier* of A, and similarly one can define a *lower frontier* $\mathcal{C}^-(A)$.

Lemma 3.7. *Under the assumptions of Lemma 3.6, we have that*

$$\partial \mathcal{C}^+(A) \;\subseteq\; \partial A. \tag{3.7}$$

In particular, any essential continuum in $\mathbb{A}$ contains the boundary of an essential circloid.

Proof. In general, when $S \subseteq \mathbb{A}$ is essential and bounded above we have $\partial \mathcal{U}^+(S) \subseteq \partial S$, and the analogous statement holds if S is bounded below. Applying this several times, we obtain

$$\partial \mathcal{U}^{+-+}(A) \;\subseteq\; \partial \mathcal{U}^{+-}(A) \;\subseteq\; \partial \mathcal{U}^{+}(A) \;\subseteq\; \partial A. \tag{3.8}$$

Furthermore, $E^+ = \partial \mathcal{U}^{+-+}(A)$ is an essential continuum which is disjoint from $\mathcal{U}^{+-}(A)$ and $\mathcal{U}^{+-+}(A)$ and therefore contained in $\mathcal{C}^+(A)$. Consequently $C^+ = \mathbb{A} \setminus (\mathcal{U}^-(E^+) \cup \mathcal{U}^+(E^+))$ is an annular continuum contained in $\mathcal{C}^+(A)$, and by minimality of the latter we obtain $C^+ = \mathcal{C}^+(A)$. However, this means that

$$\begin{aligned} \partial \mathcal{C}^+(A) &= \partial C^+ \;=\; \partial(\mathbb{A} \setminus \partial C^+) \\ &= \partial(\mathcal{U}^-(E^+) \cup \mathcal{U}^+(E^+))) \;\subseteq\; \partial E^+ \;=\; \partial \mathcal{U}^{+-+}(A) \;\subseteq\; \partial A, \end{aligned}$$

as required. □

Given two essential continua E_1 and E_2, we write $E_1 \prec E_2$ if $E_1 \subseteq \mathcal{U}^-(E_2)$. We say that a sequence of essential continua $\{E_n\}_{n \in \mathbb{N}} \subset \mathbb{A}$ is bounded if there exist two essential continua $E, F \subset \mathbb{A}$ such that $E \prec E_n \prec F$ for every $n \in \mathbb{N}$.

Lemma 3.8. *Suppose $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathbb{A}$ is a bounded sequence of essential continua with $E_n \prec E_{n+1}$ for all $n \in \mathbb{N}$. Then the E_n converge in Hausdorff limit to the essential continuum $\partial\mathcal{U}^-$, where $\mathcal{U}^- = \bigcup_{n\in\mathbb{N}} \mathcal{U}^-(E_n)$.*

Proof. We have to show that

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ :\quad \partial\mathcal{U}^- \subseteq B_\varepsilon(E_n) \text{ and } E_n \subseteq B_\varepsilon(\partial\mathcal{U}^-)\ . \tag{3.9}$$

Note that for all $n \in \mathbb{N}$ the set E_n is contained in $\mathcal{U}^-(E_k)$ for all $k > n$. Conversely, $\partial\mathcal{U}^- \subseteq \mathcal{U}^+(E_n)$ for all $n \in \mathbb{N}$, since the bounded connected components of $\mathbb{A} \setminus E_n$ are all contained in $\mathcal{U}^-(E_{n+1}) \subseteq \mathcal{U}^-$ and can therefore not intersect $\partial\mathcal{U}^-$.

Now, first assume that there exist infinitely many $n \in \mathbb{N}$ with $\partial\mathcal{U}^- \nsubseteq B_\varepsilon(E_n)$. Choose a sequence $n_i \nearrow \infty$ and $z_i \in \partial\mathcal{U}^-$ with $z_i \notin B_\varepsilon(E_{n_i})$. Note that this implies $z_i \notin B_\varepsilon(E_{n_j})$ for all $j \leq i$, since the straight arc from z_i to the nearest point in E_{n_j} first has to pass through E_{n_i}. By compactness, we may assume that the limit $z = \lim_{i\to\infty} z_i \in \partial\mathcal{U}^-$ exists. Then $B_\varepsilon(z) \cap E_n = \emptyset$ for all $n \in \mathbb{N}$. Obviously $B_\varepsilon(z)$ cannot be contained in $\mathcal{U}^-(E_n)$ for any $n \in \mathbb{N}$. Further, $B_\varepsilon(z)$ can also not be contained in a bounded component of $\mathbb{A} \setminus E_n$, since it would then be contained in $\mathcal{U}^-(E_{n+1})$. Consequently $B_\varepsilon(z) \subseteq \mathcal{U}^+(E_n)$ for all $n \in \mathbb{N}$. However, this means that $\mathcal{U}^-$ does not intersect $B_\varepsilon(z)$, contradicting $z \in \partial\mathcal{U}^-$.

Conversely, suppose $E_n \nsubseteq B_\varepsilon(\partial\mathcal{U}^-)$ for infinitely many $n \in \mathbb{N}$. Choose $n_i \nearrow \infty$ and $z_i \in E_n \setminus B_\varepsilon(\partial\mathcal{U}^-)$ so that the limit $z = \lim_{i\to\infty} z_i$ exists. Then on the one hand we have $z \notin B_{\varepsilon/2}(\partial\mathcal{U}^-)$, but on the other hand z is a limit point of points $z_i \in E_n \subseteq \mathcal{U}^-(E_{n+1}) \subseteq \mathcal{U}^-$, a contradiction. This shows that (3.9) holds and thus $\lim^{\mathcal{H}}_{n\to\infty} E_n = \partial\mathcal{U}^-$ as claimed. □

We now turn to minimal sets of type 2, starting with a simple observation.

Lemma 3.9. *Suppose $\mathcal{M}$ is a minimal set of $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}^c$ contains an essential annulus $\mathcal{A}$ of homotopy type (p,q). Then (p,q) is an eigenvector of the induced action f_* on homotopy.*

In fact, the assertion of the lemma is true for any annulus which is either invariant or disjoint from its image.

Proof of Lemma 3.9. When $\mathcal{A}$ is invariant, then the fact that its homotopy vector is preserved is obvious. When $f(\mathcal{A})$ and $\mathcal{A}$ are disjoint, this follows from the fact that essential annuli of different homotopy types have to intersect. □

Now, choose $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $(p,q) = A \cdot (1,0)^t$ and let f_A be the torus homeomorphism induced by A. Then $(1,0)$ is an eigenvector of the action on homotopy of

$$\widehat{f}\ =\ f_A^{-1} \circ f \circ f_A\ , \tag{3.10}$$

and this implies that there exists a lift $\widetilde{f} : \mathbb{A} \to \mathbb{A}$ which projects to $\widehat{f}$ under the canonical projection $\pi_{\mathbb{A}} : \mathbb{A} \to \mathbb{T}^2$. We either have $\widetilde{f}(z + (0,1)) = \widetilde{f}(z) + (0,1)$ or $\widetilde{f}(z + (0,1)) = \widetilde{f}(z) - (0,1)$. We call $\widetilde{f}$ *order-preserving* in the first case and *order-reversing* in the second. When $\widetilde{f}$ is order-preserving, we define the *rotation interval of f orthogonal to (p,q)* by

$$\rho_{(p,q)}(f)\ =\ \frac{1}{\|(p,q)\|_2} \cdot \left\{\rho \in \mathbb{R}\ \middle|\ \exists n_i \nearrow \infty,\ z_i \in \mathbb{A} : \lim_{i\to\infty} \left(\pi_2 \circ \widetilde{f}^{n_i}(z_i) - \pi_2(z_i)\right)/n_i = \rho\right\}\ . \tag{3.11}$$

Of course, due to the freedom in the choice of the lift $\widetilde{f}$ the interval $\rho_{(p,q)}$ is only well-defined up to translation by integer multiples of $\|(p,q)\|_2$, and we will implicitly understand it in this sense. Note that when f is homotopic to the identity, then $\rho_{(1,0)}(\widetilde{f})$ is just the projection of $\rho(F)$ to the second coordinate. In general, it is the projection of $\rho(F)$ to the line $(-q,p) \cdot \mathbb{Z}$.

In the order-reversing case, we apply the above definition to f^2 and let $\rho_{(p,q)}(f) = \rho_{(p,q)}(f^2)/2$. In this case, we have

Lemma 3.10. *If $\widetilde{f}$ is order-reversing, then $\rho_{(p,q)}(f)$ contains 0.*

Proof. For every $n \in 2\mathbb{Z}+1$ the map $\widetilde{f}^n$ reverses orientation, so that $D(k) = \pi_2 \circ \widetilde{f}^n(0,k) - k$ goes to $\pm\infty$ as k goes to $\mp\infty$. Consequently, for sufficiently large k the numbers $D(k)$ and $D(-k)$ have opposite sign. Therefore, by the Intermediate Value Theorem any arc joining $(0,k)$ to $(0,-k)$ contains a point with $\pi_2 \circ \widetilde{\pi}_2 \circ f^n(z) - \pi_2(z) = 0$. □

In the same way, it is shown that $\rho_{(p,q)}(f)$ is connected and, in the order-reversing case, symmetric around 0. In the situation we consider, the rotation interval is degenerate.

Lemma 3.11. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and suppose there exists an annulus $\mathcal{A} \subseteq \mathbb{T}^2$ of homotopy type (p,q) which is either periodic or wandering. Then $\rho_{(p,q)}(f)$ contains a single number. If $\mathcal{A}$ is periodic, this number is rational.*

In particular, suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has a periodic or wandering annulus. Then the rotation set $\rho(F)$ is contained in a rational line.

This is a direct corollary to [12, Lemma 1.4], and we omit the simple proof. From now on, we identify $\rho_{(p,q)}(f)$ with the unique real number ρ it contains and call it the *rotation number of f orthogonal to (p,q)*. The following lemma deals with the case where this rotation number is irrational. We omit the proof, which can be found in [12]. The author uses an additional minimality assumption, but this is actually not needed. Alternatively, the result also follows from a minor modification of [10, Proof of Theorem C].

Lemma 3.12. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ has a wandering annulus $\mathcal{A}$ of homotopy type (p,q) and $\rho_{(p,q)}(f)$ is irrational. Then f is semiconjugate to the corresponding irrational rotation on $\mathbb{T}^1$.*

In order to treat the rational case, we first need some more information concerning circloids.

Lemma 3.13. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and suppose there exists a wandering circloid $C \subseteq \mathbb{T}^2$ of homotopy type (p,q). Further, assume $\rho_{(p,q)}(f) = 0$ and let $C_0 \subseteq \mathbb{A}$ be a lift of $f_A(C)$, where $A \in SL(2,\mathbb{Z})$ is chosen as in (3.10). Then $C_0 - (0,1) \prec \widetilde{f}^{2n}(C_0) \prec C_0 + (0,1)$ for all $n \in \mathbb{N}$. In particular, the orbit of C_0 under $\widetilde{f}$ is bounded.*

Proof. Since C is wandering, $\widetilde{f}^{2n}(C)$ is disjoint from $C_0 + (0,1)\cdot\mathbb{Z}$ for all $n \neq 0$. Suppose for a contradiction that $\widetilde{f}^{2n}(C_0)$ does not lie between $C_0 - (0,1)$ and $C_0 + (0,1)$ for some $n \in \mathbb{N}$, for example $\widetilde{f}^n(C_0) \prec C_0 + (0,1)$. Then, by induction $\widetilde{f}^{i2n} \prec C_0 + (0,i)$. This, however, implies that the rotation number is strictly positive, contradicting the assumptions. □

Lemma 3.14. *Let $\widetilde{f} \in \mathrm{Homeo}(\mathbb{A})$ and suppose E is an essential continuum which is disjoint from its image and has a bounded orbit. If $\widetilde{f}$ is order-preserving, then $\lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^n(E)$ exists and contains an invariant circloid. If $\widetilde{f}$ is order-reversing, then $\lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^{2n}(E)$ exists and contains a circloid which is either invariant or two-periodic.*

Proof. It suffices to treat the order-preserving case, since we only have to consider $\widetilde{f}^2$ when $\widetilde{f}$ reverses order. We either have $E \prec \widetilde{f}(E)$ or $E \succ \widetilde{f}(E)$. We treat the first case, the other one is similar.

If we let $E_n := \widetilde{f}^n(E)$, then this is an increasing sequence with respect to $\prec$ and converges in Hausdorff distance to the essential continuum ∂U^- given by Lemma 3.8. Since $\partial U^- = \lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^n(E)$ is contained in $\mathcal{M}$ the statement follows from Lemma 3.7. □

We now turn to the proof of the fact that only essential annuli can be periodic connected components of the complement of a type two minimal set. We start with the following well-known fact.

Lemma 3.15. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and suppose $\mathcal{A} \subset \mathbb{T}^2$ is an f-invariant essential annulus. Then, for every essential simple loop γ in $\mathcal{A}$ and any neighbourhood $V \subseteq \mathcal{A}$ of γ there exists $g \in \mathrm{Homeo}(\mathbb{T}^2)$ such that:*

(i) *g is homotopic to f;*
(ii) *$g(\gamma) = \gamma$;*

(iii) *$f(x) = g(x)$ for every $x \in V^c$.*

To a homeomorphism g as in the lemma, we can naturally associate a homeomorphism $\overline{g} : \mathbb{S}^2 \to \mathbb{S}^2$, where $\mathbb{S}^2$ is the two-dimensional sphere by cutting the torus open along γ to obtain an open annulus $\mathbb{T}^2 \setminus \gamma$ and then compactifying this annulus by adding two points N and S. Then $g|_{\gamma^c}$ is conjugate to $\overline{g}|_{\{N,S\}^c}$ by a semiconjugacy $h : \mathbb{T}^2 \setminus \gamma \to \mathbb{S}^2 \setminus \{N, S\}$ and $\overline{g}(\{N, S\}) = \{N, S\}$. Furthermore, h maps the two components of $\mathrm{Cl}[\mathcal{A}] \setminus \gamma$ to two different components U_1 and U_2 in S^2 with $\mathrm{Cl}[U_1] = U_1 \cup \{N\}$ and $\mathrm{Cl}[U_2] = U_2 \cup \{S\}$. The advantage that this transformation to a sphere homeomorphism has, is that it allows to apply the following theorem by Matsumoto and Nakayama [18].

Theorem 7. *Let $\overline{g} : S^2 \to S^2$ be a homeomorphism and $C \subset S^2$ be a non singleton compact and connected set. Further assume that C is a minimal set of $\overline{g}$. Then, there are exactly two periodic connected components A_1 and A_2 in C^c.*

In our context, it is obvious from the invariance of $\mathcal{A}$ that A_1 and A_2 are the images of the two components of $\mathcal{A} \setminus \gamma$ under h. We obtain the following.

Proposition 3.16. *Suppose that $\mathcal{M}$ is a type 2 minimal set for $f \in \mathrm{Homeo}(\mathbb{T}^2)$. Further suppose there exists an essential annulus $\mathcal{A}$ in Conn$(\mathcal{M}^c)$ which is periodic. Then any disk in Conn$(\mathcal{M}^c)$ is wandering.*

Proof. We denote by $C_1, C_2 \subset \mathbb{T}^2$ the two connected components of $\partial\mathcal{A}$, allowing for $C_1 = C_2$ in case $\partial\mathcal{A}$ is connected. Then due to the fact that $\mathcal{A}$ is periodic there exists $n \in \mathbb{N}$ such that C_1 and C_2 are minimal sets of f^n and $\mathcal{M} = (C_1 \cup C_2) \cup \ldots \cup f^n(C_1 \cup C_2)$. Hence, given a disk $V \in \mathrm{Conn}(\mathcal{M}^c)$, then since ∂V is a connected set contained in $\mathcal{M}$ there exists $n_0 \in \mathbb{N}$ such that $V_1 := f^{n_0}(\partial V) \subset C_1 \cup C_2$. We assume without loss of generality that $n_0 = 0$ and $V_1 \subseteq C_1$. This means that V is a connected component of C_1^c. However, if we consider $g \in \mathrm{Homeo}(\mathbb{T}^2)$ given by Lemma 3.15 applied to f^n and some essential loop $\gamma \subseteq \mathcal{A}$, we have that C_1 is a minimal set for g.

Suppose for a contradiction that V is periodic by f. Then V is a periodic connected component of C_1^c for g. However, as V is a disk it cannot coincide with one of the two periodic components A_1 and A_2 that g admits. This contradicts Theorem 7. □

We are ready now to give the proof of the Addendum 3.5.

Proof of Addendum 3.5. Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}$ is a minimal set of type 2. Let $\mathcal{A}$ be an essential annulus of homotopy type (p, q) in $\mathrm{Conn}(\mathcal{M}^c)$. Then $\mathcal{A}$ is either wandering or periodic, and in each case $\rho_{(p,q)}(\widetilde{f})$ contains a unique number $\rho \in \mathbb{R}$ by Lemma 3.11. If ρ is irrational, then $\mathcal{A}$ is wandering by Lemma 3.11, and Lemma 3.12 provides the existence of a semiconjugacy to an irrational rotation of $\mathbb{T}^1$. Furthermore, due to the existence of such a semi-conjugacy any element in $\mathrm{Conn}(\mathcal{M}^c)$ is wandering. Thus, we are in case (ii) of the addendum.

Now, assume ρ is rational. Passing to an iterate f^k and choosing the right lift $\widetilde{f}$ of $\widehat{f}^k$ in (3.10), we may assume without loss of generality that $\rho_{(p,q)}(f^k) = 0$. Let $\mathcal{A}_0$ be a lift of $f_A(\mathcal{A})$, where $A \in \mathrm{SL}(2, \mathbb{Z})$ is chosen as in (3.10), and let $\widetilde{f}$ be the lift of f^k used to compute the rotation interval of f^k. Then either by invariance or by Lemma 3.13, the orbit of $C_0 = \mathcal{C}^+(\mathcal{A}_0)$ under $\widetilde{f}$ is bounded. Hence, by Lemma 3.14, $\lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^{2n}(C_0)$ contains an $\widetilde{f}^2$-invariant circloid $\widetilde{C}$. Since $\widetilde{C}$ is disjoint from $\mathcal{A}_0 + (0, 1) \cdot \mathbb{Z}$, it projects to a circloid C on $\mathbb{T}^2$ which is $2k$-periodic under f. Furthermore, C is contained in the Hausdorff limit of $f^{2kn}(\partial A)$ and thus in $\mathcal{M}$. By minimality, we obtain $\mathcal{M} = \bigcup_{n=1}^{2k} f^n(C)$. Moreover, in this case Proposition 3.16 implies that any disk in $\mathrm{Conn}(\mathcal{M}^c)$ has to be wandering, which means that we are in case (i) of the addendum. □

Remark 8. The above proof shows that case (i) of the addendum corresponds exactly to a rational rotation number orthogonal to the homotopy vector of the essential annuli, whereas case (ii) corresponds to an irrational rotation number.

3.3. **Minimal sets of type 3.** The following addendum to Proposition 3.1 concerning the structure of minimal sets of type 3 completes the proof of Theorem 1.

Addendum 3.17. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}$ is a minimal set of type 3. Then $\mathcal{M}$ is an extension of either a periodic orbit or a Cantor set.*

Again, we first recall some purely topological facts. We say $U \subseteq \mathbb{T}^2$ is *non-separating* if $\mathbb{T}^2 \setminus U$ is connected. We call a partition into continua $\mathcal{U} = \{U_i\}_{i\in I}$ of $\mathbb{T}^2$ an *upper semi-continuous decomposition* if it satisfies

(i) $\biguplus_{i\in I} U_i = \mathbb{T}^2$ and $U_i \cap U_j = \emptyset$ if $i \neq j$,
(ii) U_i is a compact, bounded and non-separating set for every $i \in I$;
(iii) if $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ has Hausdorff limit $\mathcal{C}$, then there exists $U_0 \in \mathcal{U}$ so that $\mathcal{C} \subset U_0$ (upper semi-continuity property).

Further, we say that a map $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ is a Moore projection for the decomposition $\mathcal{U}$ if it satisfies

(i) Φ is continuous and surjective;
(i) Φ is homotopic to the identity;
(ii) $\Phi^{-1}(x) \in \mathcal{U}$ for all $x \in \mathbb{T}^2$.

Now, the following is a classical decomposition theorem by R. Moore (see e.g.[24]).

Moore's Theorem. *For any upper semi-continuous decomposition of $\mathbb{T}^2$ there exists a Moore projection.*

For our purposes, we have to ensure that under suitable conditions this projection produces a Cantor set.

Lemma 3.18. *Let $A \subset \mathbb{T}^2$ be closed and denote its connected components by $\{A_i\}_{i\in I}$. Suppose that the decomposition $\mathcal{U} = \{A_i\}_{i\in I} \cup \{\{x\} : x \notin \bigcup_{i\in\mathbb{N}} A_i\}$ is upper semi-continuous. Then for any Moore projection Φ associated to $\mathcal{U}$, the image $\Phi(A)$ is totally disconnected.*

Proof. Suppose for a contradiction that $\Phi(A)$ is not totally disconnected, then there exists a connected component $C \subseteq \Phi(A)$ which has more than one element. Since connected components of A project to single points, the set $\Phi^{-1}(C) \subseteq A$ cannot be connected and therefore decomposes into two disjoint relatively closed subsets C_1 and C_2. As a connected component of a compact set, C is compact, and the same is true for its preimage $\Phi^{-1}(C)$. Hence, both C_1 and C_2 are compact.

For any $x \in C$, the continuum $\Phi^{-1}(x) \subseteq \Phi^{-1}(C) = C_1 \cup C_2$ is either completely contained in C_1 or completely contained in C_2. Consequently, the images $\Phi(C_1)$ and $\Phi(C_2)$ are disjoint. However, this means that C decomposes into two disjoint compact sets, contradicting its connectedness. $\square$

Finally, the following statement will be useful to verify the upper semi-continuity of decompositions.

Lemma 3.19. *Let A_n, $n \in \mathbb{N}$ be a family of compact, connected and bounded sets in $\mathbb{T}^2$. If the sets $\mathrm{Fill}(A_n)$ are pairwise disjoint and $A_n \to_{\mathcal{H}} \mathcal{A}$, then $\mathrm{Fill}(A_n) \to_{\mathcal{H}} \mathcal{A}$.*

Proof. For any fixed $\varepsilon > 0$, the fact that $A_n \to_{\mathcal{H}} \mathcal{A}$ implies that, for any fixed $\varepsilon > 0$, $\mathcal{A} \subseteq B_\varepsilon(A_n) \subseteq B_\varepsilon(\mathrm{Fill}(A_n))$ for sufficiently large n. Therefore, it suffices to show that conversely $\mathrm{Fill}(A_n) \subseteq B_\varepsilon(\mathcal{A})$ for sufficiently large n.

Suppose for a contradiction that for some $\varepsilon > 0$ there is a sequence of integers $n_k \nearrow \infty$ such that for each $k \in \mathbb{N}$ there exists some $x_k \in \mathrm{Fill}(A_{n_k}) \setminus B_{2\varepsilon}(\mathcal{A})$. Since $A_n \to_{\mathcal{H}} \mathcal{A}$, for k large enough, we have that $A_{n_k} \subset B_\varepsilon(\mathcal{A})$. By Lemma 2.7, the point x_k is contained in some disk D_k with $\partial D_k \subseteq A_{n_k}$, and for large k we have $\partial D_k \subseteq B_\varepsilon(\mathcal{A})$. Let x_0 be an accumulation point of $\{x_k\}_{k\in\mathbb{N}}$. Then $x_0 \notin B_\varepsilon(A)$, and further x_0 cannot belong to any disk D_k since these are pairwise disjoint, which follows from the assumption that the sets $\mathrm{Fill}(A_n)$ are pairwise disjoint combined with Lemma 2.7. Now, let $k \in \mathbb{N}$ such that $D_k \cap B_\varepsilon(x_0) \neq \emptyset$ and $\partial D_k \subset B_\varepsilon(\mathcal{A}) \subseteq B_\varepsilon(x_0)^c$. Then the closest point of $\mathrm{Cl}[D_k]$ to x_0 is contained in $B_\varepsilon(x_0)$ and can therefore not belong to the boundary of D_k, a contradiction. $\square$

We now finish the proof

Proof of Addendum 3.17. Let $\{\Lambda_i\}_{i\in I} := \mathrm{Conn}(\mathcal{M})$. Since $\mathcal{M}$ is of type 3, every element in $\{\Lambda_i\}_{i\in I}$ is bounded. Now, $\mathcal{U} = \{\mathrm{Fill}(\Lambda_i)\}_{i\in I} \cup \{\{x\} : x \notin \bigcup_{i\in I} \mathrm{Fill}(\Lambda_i)\}$ is a family of bounded continua in $\mathbb{T}^2$. We claim that it is an upper semi-continuous decomposition of $\mathbb{T}^2$.

First let us see that $\mathcal{U}$ is a partition. Suppose for a contradiction that $\mathrm{Fill}(\Lambda_i)\cap\mathrm{Fill}(\Lambda_j) \neq \emptyset$ for some $i \neq j \in I$. Then since $\Lambda_i\cap\Lambda_j = \emptyset$, Λ_i has to be contained in a bounded connected component D of Λ_j^c or vice versa. By Proposition 3.1, since the compact boundary of the unique doubly essential component of $\mathcal{M}^c$ is left invariant and it is contained in the minimal set $\mathcal{M}$, by minimality, $\mathcal{M}$ equals the boundary of the unique doubly essential component. If $\Lambda_i \subset D$, with D a bounded complementary domain of Λ_j^c, then an open neighborhood of a point of Λ_i does not intersect the doubly essential component, even though $\Lambda_j \subset \mathcal{M}$, a contradiction.

To proceed, the elements of $\mathcal{U}$ are non-separating. Hence, it remains to check the upper semi-continuity. For this the only non trivial case is when the sequence of elements in $\mathcal{U}$ is given by elements in $\{\mathrm{Fill}(\Lambda_i)\}_{i\in I}$. Take a countable subsequence $\{\mathrm{Fill}(\Lambda_n)\}_{n\in\mathbb{N}} \subset \{\mathrm{Fill}(\Lambda_i)\}_{i\in I}$ such that $\mathrm{Fill}(\Lambda_n) \to_{\mathcal{H}} \mathcal{X}$. Passing to a subsequence if necessary, we may assume that the Λ_n converge to a continuum $\mathcal{Y} \subseteq \mathcal{M}$. We therefore have $\mathcal{Y} \subseteq \Lambda_i$ for some $i \in I$, and by Lemma 3.19 we have that

$$\mathrm{Fill}(\Lambda_n) \to_{\mathcal{H}} \mathcal{Y} \subseteq \Lambda_i \subseteq \mathrm{Fill}(\Lambda_i). \tag{3.12}$$

Hence, $\mathcal{U}$ is upper semi-continuous.

Now, let $\Phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a Moore projection for $\mathcal{U}$. Given a point $x \in \mathbb{T}^2$, since $\mathcal{U}$ is preserved by f (i.e for every element $U_i \in \mathcal{U}$ we have $f(U_i) \in \mathcal{U}$) the set $\Phi \circ f \circ \Phi^{-1}(x)$ contains only one point $y_x \in \mathbb{T}^2$ for every $x \in \mathbb{T}^2$. Define $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ by $f(x) = y_x$. We claim that $\widetilde{f}$ is a homeomorphism. Let us prove first the continuity. For this we take $x \in \mathbb{T}^2$ and fix $\varepsilon > 0$. Then, there exists a neighbourhood V of $\Phi^{-1}(y_x)$ such that $\Phi(V) \subset B(y_x, \varepsilon)$. Moreover, there exist a neighbourhood U of $f^{-1}(\Phi^{-1}(y_x))$ such that $f(U) \subset V$. On the other hand, since $\Phi^{-1}(x) = f^{-1}(\Phi^{-1}(y_x))$ we have that there exist $\delta > 0$ such that $\Phi^{-1}(B(x, \delta)) \subset U$. Therefore, given a point $z \in B(x, \delta)$ the set $\Phi \circ f \circ \Phi^{-1}(z)$ is contained in $B(y_x, \varepsilon)$. Hence, $\widetilde{f}$ is continuous. If we define $\widetilde{f}^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\widetilde{f}^{-1}(x)$ is the unique point in $\Phi \circ f^{-1} \circ \Phi^{-1}(x)$, we have that $\widetilde{f}^{-1}$ is exactly the inverse function of $\widetilde{f}$. Moreover, by an analogous argument as above we have that $\widetilde{f}^{-1}$ is continuous. Therefore $\widetilde{f}$ is a homeomorphism.

By definition of $\widetilde{f}$, $\Phi \circ f(x) = \widetilde{f} \circ \Phi(x)$ holds for every $x \in \mathbb{T}^2$. This implies in particular that $\widetilde{\mathcal{M}} = \Phi(\mathcal{M})$ is minimal for $\widetilde{f}$. Thus, we have that $(f, \mathcal{M})$ is an extension of $(\widetilde{f}, \widetilde{\mathcal{M}})$. Moreover, Proposition 3.18 implies that $\Phi(\mathcal{M})$ is totally disconnected and therefore either a periodic orbit or a Cantor set. □

4. Special Cases and Applications

In this section, we consider several special cases of the classification and provide a number of relations of the possible minimal sets with other dynamical properties, such as the rotation set and orbit behaviour.

4.1. **Homeomorphisms homotopic to an Anosov.** To prove Corollary 2, recall classical results on Anosov diffeomorphisms.

(1) Manning [16] showed that any Anosov diffeomorphism is topologically conjugate to an algebraic Anosov, i.e. an Anosov induced by a hyperbolic element of $\mathrm{SL}(2, \mathbb{R})$.

(2) Bowen [3] showed that a minimal set of an algebraic Anosov diffeomorphism is either a periodic orbit or a Cantor set.

(3) Walters [23] provides the existence of a semiconjugacy, homotopic to the identity, between a homeomorphism homotopic to an Anosov and the underlying Anosov.

Proof of Corollary 2. If $\mathcal{M}$ would be of type 1 or 2, then by Lemma 2.6, there exists at least one unbounded connected component of $\mathcal{M}$, which we denote Λ. Let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be the

semiconjugacy between f and f_A, given by [23]. Then $\mathcal{M}' := h(\mathcal{M})$ is a minimal set of f_A and is totally disconnected due to [3, 16]. On the other hand, since h is continuous and Λ is unbounded, $h(\Lambda)$ must be an unbounded continuum in $\mathcal{M}'$, a contradiction. Since $\mathcal{M}$ can not be the whole torus either, $\mathcal{M}$ has to be of type 3. □

4.2. **Non-wandering torus homeomorphisms.** We start with three statements on periodic circloids of non-wandering torus homeomorphisms.

Lemma 4.1 ([10], Corollary 3.6)**.** *Suppose C is a periodic circloid of a non-wandering torus homeomorphism. If C does not contain periodic points, then C has empty interior.*

We call a straight line $L \subseteq \mathbb{R}^2$ *rational*, if it contains infinitely many rational points. Note that in particular, this implies that the slope of L is rational.

Lemma 4.2 ([10], Proposition 3.9)**.** *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has a periodic circloid. Then the rotation set of f is contained in a rational line.*

Given $f \in \mathrm{Homeo}(\mathbb{T}^2)$, we say an f-invariant continuum $C \subsetneq \mathbb{T}^2$ is aperiodic if it does not contain a periodic point. In [13], Koropecki identified annular continua as the only possible aperiodic invariant proper subcontinua of non-wandering torus homeomorphisms.

Theorem 4.3 ([13], Theorem 1.1)**.** *Let S be a compact orientable surface and suppose $f \in \mathrm{Homeo}(S)$ is non-wandering. Then every aperiodic invariant proper subcontinuum of S is an annular continuum.*

The following consequence will be crucial in the proof of Theorem 4 below.

Lemma 4.4. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is non-wandering. Then every periodic unbounded disk D contains a periodic point in its boundary.*

Proof. Let p be the period of D. As the boundary of an invariant open disk, ∂D is an f^p-invariant continuum. Suppose for a contradiction that ∂D does not contain a periodic point. Then it is an annular continuum A by Theorem 4.3. However, the complement of an annular continuum in $\mathbb{T}^2$ is either an open annulus $\mathcal{A}$ or the union of a punctured torus $\mathcal{T}$ and a bounded disk $\mathcal{D}$. If any of these sets contains the unbounded disk D, then by connectedness D must have further boundary points in the respective set, a contradiction. □

Proof of Theorem 4. Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ non-wandering and $\mathcal{M} \neq \mathbb{T}^2$ a minimal set for f. First, assume $\mathcal{M}$ is of type 1. Then $\mathrm{Conn}(\mathcal{M}^c)$ cannot contain bounded disks since these would have to be wandering by the Classification Theorem. Likewise, $\mathrm{Conn}(M^c)$ cannot contain an unbounded disk, since by Lemma 4.4 this unbounded disk has to contain periodic points in the boundary, and the boundary belongs to $\mathcal{M}$. Hence $\mathcal{M} = \mathbb{T}^2$. Secondly, suppose $\mathcal{M}$ is of type 2. Obviously, the essential annuli cannot be wandering, therefore $\mathcal{M}$ is equal to the orbit of the boundary of a periodic essential circloid C. However, by Lemma 4.1 the interior of C is empty, and it thus follows that $\partial C = C$.

Finally, suppose $\mathcal{M}$ is of type 3. If $\mathcal{M}$ is a Cantor extension, then for any connected component $\Lambda \in \mathrm{Conn}(\mathcal{M})$ the set $\mathrm{Fill}(\Lambda)$ is a wandering set. Therefore $\mathrm{int}(\mathrm{Fill}(\Lambda)) = \emptyset$, which means that Λ is non-separating. If $\mathcal{M}$ is a periodic orbit extension, but not a periodic orbit, then every connected component of $\mathcal{M}$ is an aperiodic invariant continuum for some iterate of f. By Lemma 4.4 it is an annular continuum, and by minimality this annular continuum must coincide with its frontiers. By Lemma 3.6 the frontiers are circloids. □

4.3. **Relations with the rotation set.** We now give a proof of the relation of the structure of minimal sets to the rotation set for homeomorphisms homotopic to the identity. We start with the proof of Corollary 3, which states that if the rotation set of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has non-empty interior, then any minimal set is of type 3.

Proof of Corollary 3. Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ have lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ and assume that the rotation set $\rho(F)$ has non-empty interior. Suppose for a contradiction that $\mathcal{M}$ is a minimal set of f that is not of type 3, that is, there exists no doubly essential component in its complement. By Lemma 3.11 the existence of an essential component in $\mathcal{M}^c$ is excluded, so

that all connected components are disks. Now, [20, Theorem A] states that for every vector $\rho \in \operatorname{int}(\rho(F))$ there exists a minimal set $\mathcal{M}_\rho$ with rotation vector ρ, that is,

$$\lim_{n\to\infty} \frac{F^n(z)-z}{n} = \rho \text{ for all } z \in \pi^{-1}(\mathcal{M}_\rho). \tag{4.1}$$

Fix $\mathcal{M}_\rho$ for some totally irrational vector $\rho \in \mathbb{R}^2$. Then, since all points in $\mathcal{M}_\rho$ are recurrent, $\mathcal{M}_\rho$ has to be contained in the orbit of a periodic disk $D \subseteq \mathcal{M}^c$. This implies that there exists a set $\mathcal{M}'_\rho \subseteq D$ which is minimal for f^p, where p is the period of D.

Choose a connected component D_0 of $\pi^{-1}(D)$, and a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f that leaves D_0 invariant. Fix $z \in \mathcal{M}'_\rho$ with lift $z_0 \in \mathbb{R}^2$ and $\delta > 0$ such that $B_\delta(z) \subseteq D$. Then irrationality of ρ together with the recurrence of z implies that there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of integers such that $\lim_{k\to\infty} f^{n_k p}(z) = z$, whereas $F^{n_k p}(z_0)$ is unbounded. Consequently, for sufficiently large $k \in \mathbb{N}$ we have that $F^{n_k p}(z_0) \subseteq B_\delta(z_0) + v$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$. However, this means that D_0 contains both z_0 and $z_0 + v$, contradicting the fact that D is homotopically trivial in $\mathbb{T}^2$. □

Finally, we turn to the proof of Corollary 5, which states that if f is a non-wandering pseudo-rotation with rotation vector ρ and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set, then

(a) if ρ is totally irrational, then $\mathcal{M}$ is an extension of a Cantor set, and
(b) if ρ is rational, then $\mathcal{M}$ is either an extension of a Cantor set, or the periodic orbit of a point or a homotopically trivial circloid.

Given a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of $f \in \operatorname{Homeo}_0(\mathbb{T}^2)$, the function $\varphi(z) = F(z) - z$ is doubly periodic and can therefore be interpreted as a function on the torus.

Theorem 4.5 ([17], Theorem 2)**.** *Suppose $f \in \operatorname{Homeo}_0(\mathbb{T}^2)$ is non-wandering and A is an essential annular continuum. Further, suppose there exists an f-invariant probability measure μ with support in A and $\rho = \int \varphi \, d\mu \in \mathbb{Q}^2$. Then there exists a periodic point in A with rotation vector ρ.*

Proof of Corollary 5. Suppose $f \in \operatorname{Homeo}_0(\mathbb{T}^2)$ is a non-wandering pseudo-rotation with rotation vector ρ and $\mathcal{M} \neq \mathbb{T}^2$ a minimal set for f.

(a) Let ρ be totally irrational. We have to rule out cases 1^{nw} and 3^{nw} in Theorem 4. First, by Lemma 4.2, $\mathcal{M}$ cannot be a union of periodic essential circloids, since these force the rotation set to be included in a rational line. Similarly, $\mathcal{M}$ cannot be a periodic orbit or a periodic orbit extension, since this implies the existence of a rational rotation vector. Note here that the factor map in the definition of a periodic orbit extension preserves rotation vectors.

(b) Let ρ be rational. In this case, we have to show that only cases 2^{nw} and 3^{nw} in Theorem 4 can occur. However, as a rational pseudo-rotation f has at least one periodic orbit[6]. Thus $\mathcal{M} \neq \mathbb{T}^2$. Further, due to Theorem 4.5 any periodic essential circloid has to contain a periodic point. This rules out case 1^{nw}. □

5. Remarks and Problems

The results in this paper give rise to a number of further problems to be elaborated upon. A recurring theme in exploring the structure of minimal sets is the existence (or not) of unbounded disks.

Problem 1 (Unbounded disks)**.** *Let $\mathcal{M}$ be a minimal set of a homeomorphism $f \in \operatorname{Homeo}(\mathbb{T}^2)$ of type 1 or 2.*

(i) *If $\mathcal{M}$ is a type 2 minimal set, is it possible to have unbounded disks in the complement of $\mathcal{M}$? Note that if if there exists some unbounded disks in $\operatorname{Conn}(\mathcal{M}^c)$, then there have to be infinitely many, since all disks are wandering by Theorem 1.*
(ii) *Do there exist rational pseudo-rotations with type 1 minimal sets?*

For recent progress concerning the problem of boundedness of invariant disks, see [14].

In Corollary 3 and 5, we considered the relation between the rotation set of a homeomorphism $f \in \operatorname{Homeo}(\mathbb{T}^2)$ and the structure of the minimal set in specific cases. In [15],

a classification was given for the non-resonant case, i.e. where the rotation set is a single totally irrational vector. Between the cases given, there is an important class of rotation sets consisting of line segments.

Problem 2 (Rotation set versus structure of minimal sets)**.** *Let $\mathcal{M}$ be a minimal set of a homeomorphism $f \in \mathrm{Homeo}(\mathbb{T}^2)$. Suppose the rotation set $\rho(f)$ is a line segment of positive length. Relate the properties of this line segment with the structure of the minimal sets the homeomorphism admits.*

For homeomorphisms homotopic to the identity, all types of minimal sets that our classification allows are realised, see [15] for these constructions. In the case where the homeomorphism is homotopic to an Anosov, the list of possible minimal sets is rather restricted, cf. Corollary 2. The case left is the class of homeomorphisms homotopic to neither the identity, nor to an Anosov, which in case of the torus are precisely the Dehn-twists. In this case, examples of type 2 as well as type 3 minimal sets are well-known to occur as minimal sets. Concerning type 1, taking a minimal Dehn-twist and blowing an orbit up to bounded disks, one obtains minimal sets for which the complement is a union of bounded disks. This leaves open one case for Dehn-twists.

Problem 3 (Unbounded disks and Dehn-twists)**.** *Is it possible for a homeomorphism homotopic to a Dehn-twist to have a type 1 minimal set with either periodic or wandering* unbounded *disks?*

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